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# THE PHYSICAL REVIEW

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## Nonlinear Meson Theory of Nuclear Forces. I. Neutral Scalar Mesons with Point-Contact Repulsion

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This paper develops an attempt to account for nuclear saturation and shell structure in terms of many-body forces that are derived from mesons that obey a nonlinear wave equation. Classical field theory is used, and in some cases the practical difficulties of obtaining numerical answers are reduced by employing a variation method. Apart from a cutoff, which appears in this particular form of the theory but could be eliminated, there are two parameters in the theory; they can be chosen so that nuclear matter has a stable density equal to the observed value, and a variational binding energy equal to 42 percent of the observed value, thus approximately accounting for saturation. The two-nucleon interaction has the observed order of magnitude in empty space, and is greatly reduced within nuclei. This suppression of two-body interactions in favor of the interaction of each nucleon with the average nucleon density in heavy nuclei may account for the independent-particle model and hence for shell structure. Although the theory does not account for magnetic moments, it indicates that a more realistic version (for example, a nonlinear pseudoscalar theory) may predict a reduction of the anomalous magnetic moments of nucleons within nuclei. According to a recent suggestion of Bloch, this could account for the deviations of the magnetic moments of even-odd nuclei from the Schmidt lines. The nonlinearity also has the consequence that mesons are scattered from nuclei as though by a strong repulsive potential. The relation of this effect to current observations on interactions between mesons and nuclei is briefly discussed.

### I. INTRODUCTION

ONE of the most fundamental and least understood properties of atomic nuclei is saturation: the close proportionality of nuclear volume and binding energy with mass number. Attempts to account for saturation in terms of two-body interactions between nucleons have taken two directions. In his first paper on the neutron-proton structure of nuclei, Heisenberg<sup>1</sup> proposed that sufficient exchange forces be introduced to account for saturation, and this idea has since been followed up extensively. Within the last few years, however, experiments on the scattering of high energy neutrons and protons, mainly at Berkeley,<sup>2</sup> have indicated quite definitely that the actual two-body exchange forces do not supply enough repulsive interaction to prevent the collapse of heavy nuclei. The second attempt to account for saturation assumed that the

two-body interaction consists of a central repulsive core surrounded by an attractive region.<sup>3</sup> Quantitative investigation showed that a repulsive core of sufficient diameter to yield the observed density of heavy nuclei would lead to disagreement with the experimental data on two-nucleon scattering.<sup>4</sup> There now appears to be general acceptance of the idea that an explanation of saturation will require the introduction of many-body forces between nucleons, in which case the potential energy of a given configuration of nuclear matter will not be determined unambiguously by the known two-body forces.

A second important property of nuclei that is not

<sup>3</sup> H. A. Bethe and R. F. Bacher, *Revs. Modern Phys.* **8**, 161 (1936); H. A. Bethe, *Elementary Nuclear Theory* (John Wiley and Sons, Inc., New York, 1947), p. 81.

<sup>4</sup> G. Parzen and L. I. Schiff, *Phys. Rev.* **74**, 1564 (1948). This question has recently been re-examined by R. Jastrow, *Phys. Rev.* **81**, 165 (1951). His conclusion that a large enough repulsive core in the singlet neutron-proton interaction could lead to saturation, cannot be accepted, since the most stable configuration of a heavy nucleus would then be a collapsed state with parallel spins (all triplet interactions); the repulsive core in the triplet interaction is much smaller if it exists at all.

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<sup>1</sup> W. Heisenberg, *Z. Physik* **77**, 1 (1932).

<sup>2</sup> See, for example, the nearly symmetric neutron-proton scattering curve obtained by R. Wallace, *Phys. Rev.* **81**, 493 (1951), and earlier papers cited there, which implies little or no interaction in odd  $l$  states (Serber force).

now adequately understood is the shell structure.<sup>5</sup> While much attention has been devoted to the details of coupling schemes that lead to particular sets of magic numbers, no quantitative study has been made of the validity of the independent-nucleon model on which these coupling schemes are based. Indeed, it appears at first that the strong, short-range two-body interactions that are known to exist between pairs of isolated nucleons would make the state of any one nucleon in a nucleus dependent mainly on the motions of its neighbors of the moment, and thus prevent the existence of the one-body potential which seems essential to shell structure. However, Wigner<sup>6</sup> has pointed out that for sufficiently short-range two-body interactions, a nucleon may not be able to follow the rapid variations in potential that are produced by the motions of its neighbors, and hence would behave as though it were in a smoothed-out version of the actual potential. Alternatively, it has been suggested that the disturbing effects of the short-range interactions on the motion of a nucleon may be suppressed by the exclusion principle, which makes it difficult for a nucleon to change its state when neighboring states are occupied. This idea, formulated most explicitly by Weisskopf,<sup>7</sup> leads in the direction of the independent-nucleon model for the lowest excited states of nuclei, and perhaps also toward the liquid-drop model for the highly excited states that follow nucleon capture. Neither of these proposed explanations for the independent-nucleon model seems to have been developed quantitatively thus far.

The present series of papers presents an attempt to account for both saturation and the independent-nucleon model in terms of a particular type of many-body force. In this theory the interactions between nucleons arise from mesons which obey a nonlinear wave equation.<sup>8</sup> Some insight into the consequences of such a model can be obtained from a preliminary qualitative discussion based on a classical treatment of the meson field in which the nucleons act as sources. In the usual linear theory, the meson field amplitude is proportional to the nucleon source strength. Meson fields are then superposable, and the interaction energy between a number of nucleons is equal to the sum of the pair interactions. In the present theory, the nonlinearity is chosen in such a way that the meson field amplitude increases less rapidly than linearly with the nucleon source strength. Then the change in meson amplitude produced by the addition of a nucleon is less when many nucleons are already present than when few are present, and the interaction energy between a number of nearby nucleons is less than the sum of the pair interactions. This is just the sort of effect needed to

account for saturation; a stable density for very heavy nuclei will be obtained if the (attractive) potential energy per nucleon increases less rapidly than the  $\frac{2}{3}$  power of the density for high densities. By the same token, the interaction between a pair of nucleons when they are embedded in a heavy nucleus (as distinguished from the interaction of each with the surrounding nuclear matter) is less than when they are in empty space. This suppression of two-body interactions within a nucleus in favor of the interaction of each nucleon with the average nucleon density, means that the nonlinearity acts as a smoothing mechanism and hence leads in the direction of the one-body potential and shell structure.<sup>9</sup>

These papers develop two ways in which a nonlinearity can be introduced into the usual meson theory of nuclear forces: in those terms that involve only the meson field and in those terms that represent the coupling between mesons and nucleons. A third possibility, placing the nonlinearity in the purely nucleonic terms, leads to meson-independent interactions between nucleons, and will not be pursued further here.<sup>10</sup> In the first case, results that are interesting from the present point of view are obtained if the nonlinearity corresponds to a repulsion between mesons. The appearance of a nonlinear term in the source-free meson equation means that it is very difficult to work with the corresponding quantum field theory. While quantization can be carried through in the usual way, it is not clear that the procedure is self-consistent, and the field energy cannot be diagonalized easily, if at all, even in the absence of nucleons. The nonlinearity can of course be treated as a perturbation, but this is a useful procedure only in special cases (see Secs. III and XI), and throws little light on the more general quantization problem. Therefore, classical field theory will be used throughout; since the field amplitudes in nuclear matter turn out to be large and the mesons obey Einstein-Bose statistics, the use of classical field theory may actually be a fairly good approximation within nuclei. In the second case, the source-free meson equation is the usual one, and can be quantized by standard methods.

The present paper deals with the first case, in which the nonlinearity is in the meson field itself. This nonlinearity can take many forms. Most of this paper is devoted to the neutral scalar meson theory in which the nonlinearity corresponds to a point-contact repulsion between mesons.<sup>8</sup> A positive term proportional to  $\phi^4$  must be added to the hamiltonian density, or a  $\phi^3$  term to the wave equation, where  $\phi$  is the meson field amplitude. While this seems a simple and natural form

<sup>5</sup> M. G. Mayer, Phys. Rev. **78**, 16 (1950), and earlier papers cited there.

<sup>6</sup> E. P. Wigner (private communication).

<sup>7</sup> V. Weisskopf, Science **113**, 101 (1951); see also H. Kopferman, Naturwiss. **38**, 29 (1951).

<sup>8</sup> L. I. Schiff, Phys. Rev. **80**, 137 (1950); Phys. Rev. **83**, 239 (1951).

<sup>9</sup> This possible relation between nonlinear meson theory and the saturation and shell structure of nuclei occurred independently to E. Teller (private communication). Similar nonlinearities have been introduced into the theory for different purposes by W. Heisenberg, Z. Naturforsch. **5a**, 251 (1950); and R. Finkelstein and M. Ruderman, Phys. Rev. **81**, 655 (1951).

<sup>10</sup> Terms of this last type also arise if the meson field is eliminated from the usual classical meson-nucleon equations; see S. D. Drell, Phys. Rev. **79**, 220 (1950).

to use, it brings a serious problem into the analysis and the interpretation of the formalism. Because of this difficulty, as many results as possible are established using a general form of nonlinearity, and the specialization to the foregoing form postponed as long as can conveniently be done. The second paper of this series, immediately following,<sup>11</sup> deals with the second case, in which the nonlinearity is put in the meson-nucleon coupling.<sup>12</sup> Since the results obtained there on the basis of classical field theory are unpromising from the point of view of explaining saturation and shell structure, it is not now planned to carry that line of approach further. It is, however, hoped that one or more further papers in this series will deal with other types of mesons and other forms of field nonlinearities, and perhaps also with the problem of quantization. In particular, a form of nonlinearity proposed by Teller<sup>13</sup> (see Eq. (7) and Sec. VI) avoids the difficulty referred to above, although it seems intuitively less natural.

The theory presented here must be regarded as no more than a model for what may eventually turn out to be a reasonably complete meson theory of nuclear forces and structure. The primary objective now is to lay the basis for forming an opinion as to whether or not the general ideas presented here have any relation to reality. For such an exploratory purpose, it seems best to make an inherently difficult analytical development as simple as possible, even at the expense of realism. This relative simplicity is achieved first by using only classical field theory, and second by choosing the mesons to be of the neutral scalar type. It is apparent, then, that results in quantitative agreement with experiment cannot be expected and that such phenomena as exchange forces and anomalous nucleon magnetic moments will not appear at all as consequences of the theory. Nevertheless, a qualitative inference concerning the latter is presented in Sec. X.

## II. FORMULATION OF THE THEORY

We choose units such that  $c$ ,  $\hbar$ , and the meson mass  $\mu$  are equal to unity. Then all lengths are measured in units of the meson Compton wavelength  $\hbar/\mu c$  ( $=1.40 \times 10^{-13}$  cm for heavy or  $\pi$ -mesons), and all energies are measured in units of the meson rest energy  $\mu c^2$  ( $=140$  Mev). The lagrangian density is assumed to have the form,

$$\mathcal{L} = \frac{1}{2}(\partial\phi/\partial t)^2 - \frac{1}{2}(\nabla\phi)^2 - G(\phi) + f(\mathbf{r}, t)F(\phi), \quad (1)$$

where  $f(\mathbf{r}, t)$  is the nucleon source density,  $F(\phi)$  the nonlinear coupling function, and  $G(\phi)$  the nonlinear field function. In the usual linear theory,

$$F(\phi) = \phi, \quad G(\phi) = \frac{1}{2}\phi^2. \quad (2)$$

We shall assume that  $F$  and  $G$  approach the forms (2)

when  $\phi$  is small, so that for weak fields the nonlinear theory becomes the usual one. The momentum canonically conjugate to  $\phi$  is  $\pi = \partial\phi/\partial t$ , the hamiltonian or energy density is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + G(\phi) - f(\mathbf{r}, t)F(\phi), \quad (3)$$

and the wave equation is

$$\partial^2\phi/\partial t^2 = \nabla^2\phi - G'(\phi) + f(\mathbf{r}, t)F'(\phi), \quad (4)$$

where a prime denotes a derivative with respect to  $\phi$ .

Some indication as to useful assumptions concerning the forms of  $F$  and  $G$  can be obtained by considering the case in which  $f(\mathbf{r}, t)$  is constant in space and time. This may be thought of as an approximation to the situation in the interior of a heavy nucleus. Then  $\phi$  is also constant, and the wave equation (4) becomes

$$G'(\phi) = fF'(\phi), \quad (5)$$

where  $f$  is proportional to the nucleon density. If now  $\phi$  is to increase less rapidly than linearly with  $f$  (see Sec. I), the ratio  $G'/F'$  must increase more rapidly than linearly with  $\phi$ . In the remainder of this paper we assume the form (2) for  $F$ , and therefore assume also that  $G$  increases more rapidly than  $\frac{1}{2}\phi^2$ . Similarly, in II we assume the form (2) for  $G$ , and also that  $F$  increases less rapidly than  $\phi$ .

With  $F(\phi) = \phi$ , Eq. (5) becomes  $G'(\phi) = f$ . It seems plausible to require that there be a unique relation between  $f$  and  $\phi$  that has no preference as to sign. We therefore assume that  $G$  is a positive even function of  $\phi$  that increases monotonically more rapidly than  $\frac{1}{2}\phi^2$ . Equation (3) gives for the energy density in this case  $G - f\phi$ , so that the energy per nucleon is proportional to  $(G/f) - \phi$ . Suppose now that for high nucleon density, when  $f$  and  $\phi$  are large,  $G$  is proportional to  $\phi^n$ , where  $n > 2$ . Then the energy per nucleon is negative and proportional to  $f^{1/(n-1)}$ . Since this energy is the average potential energy of a nucleon, and its kinetic energy increases with nucleon density as  $f^{1/2}$ , the heavy nuclear system fails to collapse in this approximation if and only if  $1/(n-1) < \frac{2}{3}$ , or  $n > 5/2$ .

Wherever detailed calculations are made in this paper, it is assumed that

$$G(\phi) = \frac{1}{2}\phi^2 + \frac{1}{4}\alpha^2\phi^4, \quad (6)$$

where  $\alpha$  is a constant to be determined later by comparison with experiment; this corresponds to  $n=4$ . Physically, the first term on the right side of Eq. (6) is the rest mass term in the usual linear theory; the second term is equivalent to a point-contact repulsion between mesons, since its space integral can be written

$$\int \phi^4 d\tau = \int \int \phi^2(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \phi^2(\mathbf{r}') d\tau d\tau'.$$

As an alternative to Eq. (6), Teller<sup>13</sup> has suggested the form,

$$G(\phi) = (1/n\alpha^2)[(1 + \alpha^2\phi^2)^{n/2} - 1], \quad (7)$$

<sup>11</sup> L. I. Schiff, Phys. Rev. **84**, 10 (1951), referred to here as II.

<sup>12</sup> This form of the theory was suggested to the writer by F. Bloch.

<sup>13</sup> E. Teller (private communication).

which has less intuitive appeal but greater flexibility than (6). We shall return briefly to Eq. (7) in Sec. VI.

### III. FREE-MESON SOLUTIONS

In the absence of sources ( $f=0$ ), Eq. (4) has solutions in the form of traveling plane waves that approach the usual harmonic plane waves in the weak field limit. To see this, we call the direction of propagation the  $x$  axis, and the wave speed  $v$  (measured in units of the speed of light). Then

$$\phi(\mathbf{r}, t) = h(x - vt),$$

and  $h$  satisfies the equation,

$$(v^2 - 1)h'' = -G'(h), \quad (8)$$

where a prime denotes differentiation with respect to the argument. Since  $G'(h)$  has the same sign as  $h$  (see Sec. II), Eq. (8) has oscillatory solutions if and only if  $v > 1$ . A first integral of Eq. (8) is easily obtained after both sides are multiplied by  $h'$ :

$$\frac{1}{2}(v^2 - 1)h'^2 = -G(h) + \text{constant}. \quad (9)$$

It is apparent from Eq. (9) that the constant of integration must be sufficiently positive if there is to be a real solution; further examination shows that for given  $v$ , the wavelength and period decrease as the wave amplitude increases, and that the form is more sharply peaked than a sinusoidal wave. When  $G$  has the form (6),  $h$  is an elliptic integral of the first kind.

For infinitesimal amplitudes, these plane wave solutions can be superposed to form wave packets that travel with the group speed  $1/v$ , as in the usual linear theory. When the amplitude is small but finite, the nonlinear term is relatively small. Then quantization can be carried through in the usual way, and the nonlinearity taken into account as a perturbation if need be. This is actually a physically interesting situation when we think of the free meson beams that are attainable in the laboratory, as we now show.

We tentatively assume that the linear theory provides a useful approximation in this case, so we can put

$$\phi(\mathbf{r}, t) = A \cos(\mathbf{k} \cdot \mathbf{r} - \omega t),$$

for a meson beam of momentum  $\mathbf{k}$  and energy  $\omega = (\mathbf{k}^2 + 1)^{1/2}$ . Setting  $f=0$  and  $G = \frac{1}{2}\phi^2$  in Eq. (3) shows that the average energy density is  $\frac{1}{2}A^2\omega^2$ , so that the number of mesons per unit volume is  $\frac{1}{2}A^2\omega$ . A very intense meson beam (probably unattainable in the laboratory) would consist of a burst of  $10^{10}$  mesons in a microsecond pulse, confined to a beam of one square millimeter cross section and traveling with a third the speed of light. This corresponds to a density of about  $3 \times 10^{-31}$  meson per Compton wavelength cubed, or an  $A$  value of about  $8 \times 10^{-16}$ . If now the nonlinearity is to be significant for nuclear structure, where the linear dimensions and energies involved are of order one when expressed in our units, the nonlinear term should only become important when  $\phi$  is of order one. This expect-

tation is confirmed by the quantitative results of Sec. VII. Thus for attainable free meson densities, the present theory is equivalent to the usual linear theory.

### IV. STABILITY OF STATIC SOLUTIONS

Because of the impossibility of separating out the time, static solutions of the wave Eq. (4) that arise from stationary source distributions  $f(\mathbf{r})$  are of unusual importance and form the subject matter of the next six sections, where they are used to calculate the energies of various configurations of nucleons. It is of interest, therefore, to see whether or not such solutions are stable in time with respect to small deviations.<sup>14</sup>

Let  $\phi_0(\mathbf{r})$  be that solution of the wave equation,

$$\nabla^2 \phi_0 - G'(\phi_0) + f(\mathbf{r}) = 0, \quad (10)$$

which obeys suitable boundary conditions.

Then if  $\phi_0(\mathbf{r})$  is changed into  $\phi_0(\mathbf{r}) + \phi_1(\mathbf{r}, t)$ , where  $\phi_1 \ll \phi_0$ , we ask whether  $\phi_1$  is oscillatory in time (stable) or increases (unstable). From Eq. (4) with  $F'=1$ , we see that  $\phi_1$  approximately satisfies the linear equation,

$$\partial^2 \phi_1 / \partial t^2 = \nabla^2 \phi_1 - \phi_1 G''(\phi_0), \quad (11)$$

where use has been made of Eq. (10) and higher powers of  $\phi_1$  have been neglected. Now the assumptions concerning the form of  $G(\phi)$  made in Sec. II are consistent with  $G''(\phi)$  being everywhere positive, and we suppose that this is always the case, as it is for both of the forms (6) and (7). Moreover,  $G''(\phi_0)$  is expected to be of order one in the neighborhood of a source, where  $\phi_0$  is appreciable (see the end of Sec. III). Thus unless  $\phi_1$  is very irregular, so that  $\nabla^2 \phi_1 / \phi_1$  is comparable with  $G''(\phi_0)$ , Eq. (11) shows that  $\partial^2 \phi_1 / \partial t^2$  has the opposite sign from  $\phi_1$ , and the solution  $\phi_0$  is stable. In any event, further study of the time dependence of  $\int \phi_1 d\tau$  over various regions of space shows that any instability due to the largeness of  $\nabla^2 \phi_1 / \phi_1$  is likely to be limited both in space and in time.

### V. VARIATION PRINCIPLE FOR STATIC SOLUTIONS

Because of the great difficulty in solving nonlinear differential equations in all but the simplest cases, it is important to have a variation principle available for the estimation of the energies associated with various source distributions. We now show that in the static case, the negative of the lagrangian, computed with the correct source function and an arbitrary trial wave function, gives an upper limit on the energy, and has a stationary value equal to the correct energy when the trial function is in the infinitesimal neighborhood of the correct wave function.

For the source function  $f(\mathbf{r})$ , the correct wave function  $\phi_0(\mathbf{r})$  is a regular solution of the wave Eq. (10); the correct total energy is the integral of Eq. (3) and can

<sup>14</sup> The desirability of investigating this point was suggested to the writer by S. D. Drell.

be written in one of the equivalent forms:

$$H_0 = \int [\tfrac{1}{2}(\nabla\phi_0)^2 + G(\phi_0) - f(\mathbf{r})\phi_0] d\tau, \quad (12)$$

$$H_0 = \int [-\tfrac{1}{2}(\nabla\phi_0)^2 + G(\phi_0) - \phi_0 G'(\phi_0)] d\tau, \quad (13)$$

$$H_0 = \int [G(\phi_0) - \tfrac{1}{2}\phi_0 G'(\phi_0) - \tfrac{1}{2}f(\mathbf{r})\phi_0] d\tau. \quad (14)$$

Equations (13) and (14) are obtained from Eq. (12) by use of the wave equation and partial integration, assuming that  $\phi_0$  obeys boundary conditions that make the resulting surface integral vanish. With the trial function  $\phi^{(v)}(\mathbf{r})$ , the variation expression is minus the integral of Eq. (1):

$$H^{(v)} = \int [\tfrac{1}{2}(\nabla\phi^{(v)})^2 + G(\phi^{(v)}) - f(\mathbf{r})\phi^{(v)}] d\tau. \quad (15)$$

We now put  $\phi^{(v)}(\mathbf{r}) = \phi_0(\mathbf{r}) + \phi_1(\mathbf{r})$ , where  $\phi_1$  is not necessarily small. Substitution into Eq. (15) yields, with the help of Eqs. (10) and (12) and partial integration,

$$H^{(v)} = H_0 + \int [\tfrac{1}{2}(\nabla\phi_1)^2 + G(\phi_0 + \phi_1) - G(\phi_0) - \phi_1 G'(\phi_0)] d\tau. \quad (16)$$

Now we have assumed that  $G(\phi)$  is a positive, even, monotonic increasing function of  $\phi$  that is everywhere concave upward ( $G''(\phi)$  everywhere positive). Then  $G(\phi_0) + \phi_1 G'(\phi_0)$  regarded as a function of  $\phi_1$  is a straight line that is tangent to the curve  $G(\phi_0 + \phi_1)$  at the point  $\phi_1 = 0$  and lies below it everywhere else. It follows that the integral on the right side of Eq. (16) is positive or zero, so that  $H^{(v)}$  provides an upper limit on  $H_0$ . Moreover,  $H^{(v)} = H_0$  if and only if  $\phi_1$  is everywhere zero so that  $\phi^{(v)} = \phi_0$ ;  $H^{(v)}$  differs from  $H_0$  by terms of second order in  $\phi_1$  when  $\phi_1$  is small, so that  $H^{(v)}$  is stationary when  $\phi^{(v)}$  is in the infinitesimal neighborhood of  $\phi_0$ . It is interesting to note that the assumed properties of the nonlinear function  $G(\phi)$  are decisive in making  $H^{(v)}$  an upper limit on  $H_0$ , although they do not affect the stationary character of  $H^{(v)}$ .

## VI. SOLUTION FOR AN ISOLATED NUCLEON AT REST

It is interesting to inquire first whether there can be a regular, localized solution of Eq. (10) in the absence of sources ( $f=0$ ). If we assume the solution to be spherically symmetric, it is convenient to put  $\phi_0(\mathbf{r}) = \chi(r)/r$ , when  $\chi(r)$  satisfies the equation,

$$d^2\chi/dr^2 = rG'(\chi/r). \quad (17)$$

Now  $G'(\chi/r)$  has the same sign as  $\chi$ , so that  $d^2\chi/dr^2$  also has the same sign as  $\chi$ . This means that if  $\chi$  vanishes at infinite distance and starts to increase as  $r$  decreases from infinity, it is always concave upward

and continues to increase into the origin. Then  $\chi(r)$  cannot be zero at  $r=0$ , and  $\phi_0$  is irregular at the origin. We therefore conclude that mesons cannot be permanently localized in the absence of sources. This result is a consequence of the assumed properties of  $G(\phi)$ , which correspond to a repulsion between mesons (this repulsion was pointed out explicitly in the discussion of Eq. (6), and follows whenever  $G$  increases more rapidly than  $\phi^2$ ). If  $G$  had been chosen to correspond to an attraction between mesons, regular localized solutions could exist, but the theory would not account for saturation.

We now consider the possibility that Eq. (10) has regular, localized solutions when  $f(r)$  is a spherically symmetric source of arbitrarily small spatial extension (that is, approaching a point source). We suppose that  $f(r)=0$  for  $r>a$ , so that Eq. (17) is valid in this region. Then  $\chi(r)$  increases as  $r$  decreases, and reaches the value  $\chi(a)$  at  $r=a$ . For  $r<a$ , Eq. (17) is replaced by

$$d^2\chi/dr^2 = rG'(\chi/r) - rf(r). \quad (18)$$

If now  $\chi(a)$  has the opposite sign from  $f$ , which is assumed for simplicity to have the same sign throughout the interior of the sphere  $r=a$  (so that it approaches a simple or  $\delta$  function point source), Eq. (18) shows that  $\chi$  continues to increase into the origin, so that  $\phi_0$  is irregular there. We therefore consider only the case in which  $\chi(a)$ , and hence  $\chi(r)$  for  $r>a$ , has the same sign as  $f$ . By a repetition of the same argument,  $\phi_0$  is irregular at the origin if  $\chi$  changes sign between  $r=a$  and  $r=0$ . Thus the only way in which a regular solution  $\phi_0$  can exist is for  $\chi(r)$  to have the same sign everywhere and vanish at  $r=0$ .

We suppose without loss of generality that  $f$  and  $\chi$  are everywhere positive. We then multiply Eq. (18) through by  $4\pi r$ , integrate over  $r$ , and obtain an expression for the total source strength:

$$g = 4\pi \int_0^\infty r^2 f(r) dr = 4\pi \int_0^\infty r^2 G'(\chi/r) dr - 4\pi \int_0^\infty r (d^2\chi/dr^2) dr.$$

The last integral is easily evaluated, and vanishes if  $\phi_0$  is regular. We are interested in the point-source limit of vanishingly small  $a$ . Then since  $\chi(r)$  becomes arbitrarily large as  $r$  becomes arbitrarily small (but greater than  $a$ ), we choose  $a$  small enough so that there exists a fixed radius  $b$  greater than  $a$  for which  $G'(\phi)$  has attained its asymptotic form  $C\phi^{n-1}$  (see Sec. II). We can then obtain a lower limit for  $g$ :

$$\begin{aligned} g &= 4\pi \int_0^\infty r^2 G'(\chi/r) dr \geq 4\pi \int_a^b r^2 G'(\chi/r) dr \\ &= 4\pi C \int_a^b (\chi^{n-1}/r^{n-3}) dr \geq 4\pi C \chi^{n-1}(b) \int_a^b dr/r^{n-3}. \end{aligned}$$

The last integral converges in the limit  $\alpha \rightarrow 0$  if and only if  $n-3 < 1$ , or  $n < 4$ . Thus for a simple or  $\delta$ -function point source, the source strength  $g$  is infinite if  $n \geq 4$ . We interpret this to mean that there is no admissible solution for a point source in this case.

It follows that for the  $G(\phi)$  given by Eq. (6), a nucleon must be represented by a source of finite extent, so that the size and shape of the nucleon source appear in the theory as independently variable parameters, along with  $\alpha$  and  $g$ . The number of the independent parameters that appear in the theory can be reduced by adopting the form (7) for  $G(\phi)$ , with  $5/2 < n < 4$ ; with this range of  $n$ , the theory is expected to lead to saturation (see Sec. II) and nucleons can be represented by point sources. With such point interactions, there is also the hope of eventually developing a relativistically covariant quantum theory.

Since the use of Eq. (6) demands a nucleon of finite size, and there is no *a priori* preference as to shape, we choose an isolated nucleon source function for which the wave equation is easily solved, and which is as small in spatial extent as can conveniently be found. Such a source function can be constructed by choosing a suitable wave function  $\phi_1(r)$ , and using the wave equation,

$$\nabla^2 \phi_1 - \phi_1 - \alpha^2 \phi_1^3 + f_1 = 0, \quad (19)$$

to determine the corresponding source  $f_1(r)$ . Our choice is

$$\phi_1(r) = \frac{A}{r} (e^{-r} - e^{-\gamma r}), \quad (20a)$$

$$f_1(r) = \frac{A}{r} (\gamma^2 - 1) e^{-\gamma r} + \frac{\alpha^2 A^3}{r^3} (e^{-r} - e^{-\gamma r})^3, \quad (20b)$$

where  $\gamma$  is a parameter that is later allowed to approach infinity; in this limit:

$$\phi_1(r) \rightarrow (A/r) e^{-r}, \quad (21a)$$

$$f_1(r) \rightarrow 4\pi A \delta(r) + \frac{\alpha^2 A^3}{r^3} e^{-3r}, \quad (21b)$$

so that  $f_1$  consists of a point source surrounded by a quite small region in which the source strength is finite.<sup>16</sup>

This choice is convenient so far as the subsequent analysis and the smallness of the source are concerned. However, it is inconvenient as regards the fact that  $g$  becomes infinite as  $\gamma \rightarrow \infty$ , so that a cutoff is needed to keep the source strength finite. Since the analysis is simplified by taking the limit  $\gamma \rightarrow \infty$ , we evaluate  $g$  by choosing a radius equal to the Compton wavelength of the proton ( $=0.150$  in our units), and arbitrarily replacing the second term on the right side of Eq. (21b)

for all smaller radii by its value at  $r=0.150$ . This gives

$$g = 4\pi A + 4\pi \alpha^2 A^3 [-\text{Ei}(-0.450) + \frac{1}{3} e^{-0.150}] = 4\pi A (1 + 0.837 \alpha^2 A^2). \quad (22)$$

Since the actual divergence in  $g$  is only logarithmic, the precise nature of the cutoff is relatively unimportant. This is the only place in the theory at which a cutoff need be introduced, since while all energies diverge like  $\gamma$  for large  $\gamma$ , the divergent parts can be subtracted without ambiguity. The self-energy of the nucleon represented by Eqs. (20) can be found from Eqs. (12), (13), or (14); in the limit of large  $\gamma$  it becomes

$$H_1 = -4\pi A^2 \left\{ \frac{1}{4} \gamma - \frac{1}{2} + 3\alpha^2 A^2 [\gamma \ln(32/27) - \ln(3\gamma/32) - 1] \right\}. \quad (23)$$

## VII. MODEL FOR NUCLEAR MATTER

In order to investigate the predictions of the present theory with regard to the saturation of nuclei, it is desirable to choose as simple a model as possible. We consider a representative sample of the interior of a heavy nucleus, and regard this as typical of a large amount of nuclear matter, thus avoiding boundary effects. The binding energy per nucleon of this material is known experimentally to be about 14 Mev,<sup>16</sup> or 0.10 in our units. This value is the volume term in the mass defect; the neutron excess, coulomb, and surface terms are ignored here because we deal with an equal mixture of neutrons and uncharged protons, with no boundaries present. If we accept the experimental evidence for the independent-nucleon model, the kinetic energy per nucleon can be calculated on the basis of the Fermi gas model,<sup>17</sup> and is equal in our units to

$$\frac{3}{40} (9\pi)^{\frac{1}{3}} \frac{0.150}{r_0^2} = \frac{0.1046}{r_0^2}, \quad (24)$$

where 0.150 is the Compton wavelength of the proton and  $4\pi r_0^3/3$  is the volume per nucleon. The problem then is to calculate the potential energy per nucleon, and show that as  $r_0$  is varied, the sum of it and (24) has a minimum value equal to  $-0.10$  when  $r_0$  has its experimental value 1.0. Thus the two experimental parameters of nuclear density and binding energy per nucleon serve to determine the two parameters  $\alpha$  and  $g$ , or  $\alpha$  and  $A$ , of the theory, once the cut-off is introduced as in Sec. VI.

Two specific calculations of the potential energy are made in this section. The first is modeled after the discussion of Sec. II, according to which  $f$  and  $\phi$  are assumed to have the constant values  $f_N$  and  $\phi_N$ , related by the wave equation,

$$\phi_N + \alpha^2 \phi_N^3 = f_N. \quad (25)$$

<sup>16</sup> A larger and less singular source function has also been investigated by S. D. Drell (private communication), with results similar to those quoted in Sec. VII.

<sup>17</sup> L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, Inc., New York, 1948), p. 24.

<sup>18</sup> L. Rosenfeld, see reference 16, p. 193.

We assume that all nucleons have the same sign of  $g$ , and that their sources superpose, so that

$$f_N = \frac{3g}{4\pi r_0^3} = \frac{3A}{r_0^3} (1 + 0.837\alpha^2 A^2). \quad (26)$$

The potential energy per nucleon is then given by Eqs. (12), (13), or (14) when the integral is carried over the volume of one nucleon:

$$H_N = -\frac{4\pi r_0^3}{3} \left( \frac{1}{2}\phi_N^2 + \frac{3\alpha^2}{4}\phi_N^4 \right). \quad (27)$$

Elimination of  $f_N$  between Eqs. (25) and (26) gives a relation between  $\phi_N$  and  $g$  or  $A$ ; the sum of Eqs. (24) and (27) gives a relation between binding energy,  $r_0$ , and  $\phi_N$ ; and minimization of the binding energy with respect to  $r_0$  gives a relation between  $r_0$ ,  $\phi_N$ , and  $g$ . All three of these relations involve  $\alpha$ , so that when the experimental values of binding energy and  $r_0$  are inserted into them, they can in principle be solved for the three unknowns  $\alpha$ ,  $\phi_N$ , and  $g$ . Alternatively, if  $\alpha$  and  $g$  are determined in some other way, the three relations can be solved for binding energy,  $r_0$ , and  $\phi_N$ .

The second calculation of the potential energy per nucleon is somewhat more realistic. A nucleon with source distribution  $f_1$  is embedded in nuclear matter represented by the constant source distribution  $f_N$ . The combined source distribution is assumed to be  $f_1 + f_N$ , and the two terms are supposed to have the same sign (so that all nucleons are equivalent sources). The difference in energy between the combined source and the separated sources is then found. This calculation takes the independent-nucleon model quite literally, since it is assumed that there is no correlational change of  $f_N$  in the neighborhood of  $f_1$ ; it is difficult to estimate the sign of this effect, although it does not seem to be large. The wave equation for the combined source cannot be solved analytically, and the variation principle of Sec. V is employed. This overestimates the potential energy, so that the calculated magnitude of the binding energy per nucleon is expected to be less than the experimental value.

The question immediately arises as to whether or not the terms proportional to  $\gamma$  and  $\ln\gamma$  in the nucleon self energy expression (23) will cancel with the analogous terms in the energy of the combined source. This question can be answered by considering the exact solution of the wave equation,

$$\nabla^2\phi_{1N} - \phi_{1N} - \alpha^2\phi_{1N}^3 + f_1 + f_N = 0,$$

in the neighborhood of the origin. With the substitution  $\phi_{1N} = \chi(r)/r$ , the equation becomes

$$\frac{d^2\chi}{dr^2} - \chi - \frac{\alpha^2\chi^3}{r^2} + rf_1 + rf_N = 0.$$

In the limit  $\gamma \rightarrow \infty$ , the  $\delta$ -function in Eq. (21b) can be

taken into account by requiring that  $\chi(0) = A$ . We have then to solve the equation,

$$\frac{d^2\chi}{dr^2} - \chi - \frac{\alpha^2\chi^3}{r^2} + \frac{\alpha^2 A^3}{r^2} e^{-\gamma r} + rf_N = 0, \quad (28)$$

in the neighborhood of the origin; this can be done by using the power series:

$$\chi(r) = A[1 + a_1 r + a_2 r^2 + \dots + r^s(b_0 + b_1 r + b_2 r^2 + \dots)], \quad s > 0.$$

Substitution into Eq. (28) shows that  $a_1 = -1$ , and that  $s$  satisfies the equation,

$$s(s-1) = 3\alpha^2 A^2;$$

this has just one positive root, and it is greater than unity. The quantity  $b_0$  is arbitrary, and is to be chosen in such a way as to satisfy the boundary condition at infinity:  $\chi(r) \rightarrow r\phi_N$ . Thus the  $1/r$  and constant terms in the series expansion of  $\phi_{1N}$  are the same as for  $\phi_1$ .

The cancellation of the infinite self-energy terms is now most easily seen by using Eq. (14), according to which

$$H_{1N} = - \int [\frac{1}{4}\alpha^2\phi_{1N}^4 + \frac{1}{2}(f_1 + f_N)\phi_{1N}] d\tau. \quad (29)$$

In the limit  $\gamma \rightarrow \infty$ , the infinite terms are of two types, proportional to  $\gamma$  and to  $\ln\gamma$ . The first type arises from integrals of the forms,  $\int dr/r^2$  and  $\int \delta(r) dr/r$ ; the second type arises from integrals of the form,  $\int dr/r$ . It is easily seen from Eqs. (21b) and (29) that all of these divergent integrals are exactly the same in the expressions for  $H_{1N}$  and for  $H_1$ , since they depend only on  $f_1$  and on the  $1/r$  and constant terms in  $\phi_{1N}$  and  $\phi_1$ . Thus the potential energy per nucleon is finite, and can be calculated by subtracting  $H_1$  and  $H_N$  from  $H_{1N}$ , when the integrals for the latter two quantities are extended over equal large volumes of nuclear matter.

We now calculate  $H_{1N}$  by the variation method. A convenient trial function to use is

$$\phi_{1N}^{(v)} = \frac{A}{r} (e^{-\beta r} - e^{-\gamma r}) + \phi_N, \quad (30)$$

where  $\beta$  is the variation parameter. It is apparent that unless  $\beta = 1 + (\phi_N/A)$ , the constant term in the series expansion of  $\phi_{1N}$  is not the same as that for  $\phi_1$  in the limit  $\gamma \rightarrow \infty$ . It might be thought at first that this would make the calculated potential energy diverge like  $\ln\gamma$ ; actually, because of the form of the variation integral (15), this is not the case, as is readily verified. On the other hand, if we were to replace  $A$  by  $A'$  and  $\gamma$  by  $\gamma'$  in Eq. (30), divergences would appear; indeed, it can be shown that the leading terms in the variational energy for large  $\gamma$  and  $\gamma'$  are minimized by setting  $A' = A$  and  $\gamma' = \gamma$ , and that in this case the divergent terms cancel with those in  $H_1$ . We therefore use the



trial function (30), and find that the variation energy given by Eq. (15) yields for the potential energy per nucleon (in the limit  $\gamma \rightarrow \infty$ ):

$$H_{1N}^{(v)} - H_1 - H_N = 4\pi \left\{ \frac{1}{4} [A^2(\beta-1)^2/\beta] - A\phi_N \right. \\ \left. + \alpha^2 A^4 [\beta \ln(4\beta) - (3+\beta) \ln(3+\beta) + 6 \ln 2] \right. \\ \left. + (3\alpha^2 A^2 \phi_N^2/4\beta) - 3\alpha^2 A^3 \phi_N \ln \beta \right\}. \quad (31)$$

As  $\beta$  is varied, the minimum value of Eq. (31) always occurs for  $\beta > 1$ .

The minimum value of Eq. (31) is now used as the potential energy per nucleon to calculate the values of the theoretical parameters as outlined just below Eq. (27). If we assume  $r_0 = 1.00$ , we find that we cannot make the total (potential plus kinetic) energy per nucleon as low as  $-0.10$ . The lowest value obtained is  $-0.042$ , for which the theoretical parameters have the following values:  $g = 1.49$ ,  $A = 0.0854$ ,  $\alpha = 7.96$ ,  $\phi_N = 0.149$ ,  $f_N = 0.356$ ,  $\beta = 2.50$ . It is of incidental interest to make use of these values for  $g$  and  $\alpha$ , in connection with Eqs. (24), (25), (26), and (27), to calculate the potential and total energy by the somewhat cruder first method. It turns out that the total energy per nucleon is a minimum when  $r_0 = 1.03$ , and then has the value  $-0.020$ ; the other parameters are:  $\phi_N = 0.143$ ,  $f_N = 0.324$ . Note that either of these methods could be used to calculate the compressibility of nuclear matter.

When account is taken of the simplicity of the model (neutral scalar mesons, all nucleons equivalent), and of the use of the variation method, the foregoing results may be regarded as showing sufficient promise to warrant further development of the underlying theoretical ideas, as well as the further application of the present model that are presented in the next four sections.

## VIII. TWO-NUCLEON INTERACTION IN EMPTY SPACE

The theoretical parameters obtained in the preceding section should be consistent with the known two-nucleon interactions. For this system, the source function is

$$f_2(\mathbf{r}) = f_1(\mathbf{r} - \mathbf{r}_1) + f_1(\mathbf{r} - \mathbf{r}_2),$$

where  $f_1$  is given by Eq. (20b), and  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are the coordinates of the two nucleons. This again assumes that the sources are superposable and that all nucleons have the same sign of source. The variation method is used, and the trial function taken to be

$$\phi_2^{(v)}(\mathbf{r}) = \phi_1(\mathbf{r} - \mathbf{r}_1) + \phi_1(\mathbf{r} - \mathbf{r}_2), \quad (32)$$

where  $\phi_1$  is given by Eq. (20a). A variational parameter could be introduced into Eq. (32) by replacing  $\exp(-r)$  by  $\exp(-\epsilon r)$  in Eq. (20a); however, the minimization with respect to  $\epsilon$  requires more numerical work than seems worth while at this stage. It is expected that Eq. (32) will give best results for the energy when the separation distance  $R$  of the nucleons is large and the overlap of the two parts of  $\phi_2$  is small, and poor results

when the nucleons are close together. This turns out to be the case, as is shown below.

The argument presented in Sec. VII for the cancellation of the infinite self-energy is easily extended to include the present situation, since near  $\mathbf{r} = \mathbf{r}_1$ , the source centered at  $\mathbf{r} = \mathbf{r}_2$  has the same kind of effect that the uniform source  $f_N$  had in the discussion of the preceding section. The potential energy of interaction between the two nucleons is then the variational energy calculated from Eq. (32), less twice the nucleon self-energy given by Eq. (23). The result is, in the limit  $\gamma \rightarrow \infty$ ,

$$-4\pi A^2 \frac{e^{-R}}{R} + \frac{6\pi \alpha^2 A^4}{R} \int_1^\infty e^{-2Rx} \ln \left( \frac{x+1}{x-1} \right) \frac{dx}{x}. \quad (33)$$

The first term of Eq. (33) is just the usual attractive Yukawa interaction that is obtained from linear neutral scalar mesons, by either classical or quantum theory. The second term is repulsive, and falls off very rapidly for large  $R$ , like  $[(\ln R) \exp(-2R)]/R^2$ .

With the theoretical parameters obtained at the end of Sec. VII, the potential energy (33) is strongly repulsive near  $R=0$ . This result shows the poorness of the trial function (32) in this region, since a better trial function gives an attraction there.<sup>18</sup> With  $R=0$ , we have  $f_2(\mathbf{r}) = 2f_1(\mathbf{r})$ , and we choose a new trial function to be  $C\phi_1(\mathbf{r})$ . Variation of the parameter  $C$  shows that the interaction potential energy is negatively infinite. A comparison between the form of the divergence in this case and in the linear theory case as  $\gamma \rightarrow \infty$  shows that the sign and magnitude of the interaction near  $R=0$  are similar to those of the first (Yukawa) term of Eq. (33). We thus conclude that the true potential energy is not likely to be greatly different from this Yukawa term for all  $R$ . With  $A = 0.0854$ , this has the order of magnitude of the observed two-nucleon interactions.

## IX. TWO-NUCLEON INTERACTION IN NUCLEAR MATTER

As discussed in Sec. I, nuclear shell structure can be understood if the interaction between two nucleons embedded in nuclear matter is substantially less than their interaction in empty space. The variational calculation is based on a two-center trial function similar to Eq. (30), and the result is even more complicated than in the preceding section; only the leading term of the interaction potential energy for large  $R$  is quoted:

$$\left[ -\frac{2\pi A^2(\beta^2-1)}{\beta} + \frac{6\pi \alpha^2 A^2 \phi_N^2}{\beta} \right] e^{-\beta R}.$$

Since  $\beta = 2.50$ , this falls off much more rapidly than (33) for large  $R$ . Numerical comparison shows that the ratio of interaction energy within the nucleus to that in

<sup>18</sup> The writer is indebted to G. Breit and R. P. Feynman for discussion of this point.

empty space is about 0.07 at  $R=2$ , 0.15 at  $R=1.5$ , and 0.30 at  $R=1$ . This indicates a substantial suppression of the two-nucleon interaction within nuclear matter, at least in the domain of  $R$  where the two calculations are valid.

#### X. MAGNETIC MOMENT SUPPRESSION WITHIN NUCLEI

Bloch<sup>19</sup> has recently suggested that the deviations of the magnetic moments of even-odd nuclei from the Schmidt lines be explained in terms of a reduction of the anomalous magnetic moment of the odd neutron or proton as compared with its empty-space value. He finds that the reduction averages about a factor of two for the heavier elements, with fluctuations that correlate to some extent with the known magic numbers.

The type of theory developed here predicts an effect of this kind, since the self mesonic field of a nucleon is expected to be much less when the nucleon is embedded in nuclear matter than when it is in empty space. While the detailed theoretical model of the present paper is too simple to account for magnetic moments, it is nevertheless tempting to speculate that the factor  $\beta$  ( $=2.5$ ) which appears in the exponent of Eq. (30) but not in Eq. (20a) corresponds to an anomalous magnetic moment reduction by a factor of order  $\beta$ . A definitive statement on this point will however have to await the development of, for example, a nonlinear pseudoscalar theory.

#### XI. INTERACTION OF MESONS WITH NUCLEI

An interesting consequence of a nonlinear meson theory is that a free meson in the neighborhood of a nucleus will be strongly affected by the relatively large static meson amplitude present within the nucleus. To study this effect, we consider the time-dependent wave equation for a static source  $f_N$ , to which corresponds a static wave function  $\phi_N$ ;  $f_N$  and  $\phi_N$  are approximately constant and equal to the values quoted in Sec. VII within the nucleus, and vanish outside. The time-dependent wave equation has a solution of the form  $\phi_N(\mathbf{r}) + \phi_F(\mathbf{r}, t)$ ; if  $\phi_F$  is a small-amplitude wave, it satisfies to good approximation the linear wave equation,

$$\partial^2 \phi_F / \partial t^2 = \nabla^2 \phi_F - \phi_F - 3\alpha^2 \phi_N^2 \phi_F, \quad (34)$$

which can be derived in precisely the same way as Eq. (11).

Equation (34) can be interpreted as showing that nuclear matter acts as a strongly repulsive potential for small-amplitude meson waves in the vicinity. The strength of this equivalent repulsion is conveniently specified in terms of the distance in which the amplitude of an incident meson wave of unit energy is decreased by a factor  $e$ . This distance is  $(3\alpha^2 \phi_N^2)^{-1} = 0.487$  in our units, or  $0.68 \times 10^{-13}$  cm.

This result implies that incident mesons will be scattered by a heavy nucleus as though it were a nearly impenetrable sphere. Note, however, that for an isolated nucleon,  $\phi_N^2$  is replaced by  $A^2 \exp(-2r)/r^2$  from Eq. (21a), which gives quite a small scattering volume. The related processes of inelastic scattering, absorption, and production (by photons or energetic nucleons) are not so easily discussed, since they must be based on quantum theory. Fortunately, it is in just this case of small-amplitude waves that a quantum field theory can be made (see Sec. III), so that one might hope to go further by quantizing  $\phi_F$  but not the static meson field  $\phi_N$ . So far as meson production in heavy nuclei is concerned, it follows from the foregoing discussion that the outgoing meson wave is much more strongly coupled to the surface than to the interior of the nucleus, so that most mesons will be produced in a surface layer about  $10^{-13}$  cm thick. An effect like this seems to have been observed,<sup>20</sup> although it is doubtful if the layer is actually as thin as predicted here.

#### XII. CONCLUSIONS

The nonlinear meson theory presented in this paper, which describes neutral scalar mesons with point-contact repulsion, is certainly an oversimplification of the actual situation, and possesses the inherent difficulty of requiring finite nucleon sources. Nevertheless, it gives results that can be related in a sensible way with experimental observation. After the cut-off is fixed at the proton Compton wavelength, there are two free parameters in the theory: the nonlinear parameter  $\alpha$ , and the nucleon source strength  $g$ . These can be chosen so that nuclear matter has a stable density equal to the observed value, and a binding energy (calculated by the variation method) equal to 42 percent of the observed value. The two-nucleon interaction then comes out to have the observed order of magnitude in empty space. Within nuclei, the two-nucleon interaction is strongly reduced, and provides a qualitative explanation for nuclear shell structure. The self-mesonic field of a nucleon within a nucleus is much smaller in spatial extent than it is in empty space; this may account for the observed deviations of magnetic moments of even-odd nuclei from the Schmidt lines by making the anomalous nucleon magnetic moments smaller when they are within nuclei than when they are in empty space.<sup>19</sup> For attainable free meson beams, the density is so small that the nonlinearity is not significant. When mesons interact with a nucleus, however, the large meson amplitude in and near the nucleus acts through the nonlinearity to produce an effective repulsive potential. This effect may be significant in connection with current observations on the scattering, absorption and production of mesons in the vicinity of nuclei.

<sup>19</sup> F. Bloch, Phys. Rev. **83**, 1062 (1951). Note added in proof:—H. Miyazawa, Prog. Theor. Phys. **6**, 263 (1951), has made the same suggestion independently, and proposed also that the magnetic moment suppression may be related to the exclusion principle.

<sup>20</sup> R. F. Mozley, Phys. Rev. **80**, 493 (1950); R. M. Littauer and D. Walker, Phys. Rev. **82**, 746 (1951); Panofsky, Steinberger, and Steller, Phys. Rev., to be published.